Einstein and Bell, von Mises and Kolmogorov: reality and locality, frequency and probability

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Abstract

We perform analysis of Bell's arguments (and their generalizations) on the basis of the frequency approach to probability theory (R. von Mises, 1919). This analysis demonstrated that there are no physical arguments which support the existence of 'probability distributions' which were formally used in all investigations based on the conventional (Kolmogorov, 1933) approach to probability theory. In fact, probability distributions may fluctuate from run to run (no collectives). This chaos on the microlevel need not contradict to the stabilization of frequencies for physical observables. We discuss the relation to the completeness of quantum mechanics (and the efficiency of detectors). If micro-reality is also statistically stable (as the macro-reality), then the only root of the violation of Bell's inequality (or CHSH inequality) is the dependence of collectives corresponding to two different measurement devices. Such a dependence implies the violation of the factorization rule for the simultaneous probability distribution. Formally this rule coincides with the well known BCHS locality condition (or outcome independence condition). However, von Mises' approach implies completely different interpretation of dependence. It is not a dependence of events, but it is a dependence of collectives (merely via the same preparation procedure).

1 Introduction

Last years became more evident that Bell's arguments (which imply Bell's inequality [1] and its generalizations [2], [3]) are closely related to foundations of probability theory, see [4]- [7]. Despite the general viewpoint [1]-[3] that experimental violations of Bell's inequality [2] imply the impossibility to use the *local realism* in quantum theory, there are many publications [4]-[7] in that it was pointed out that the derivation of Bell's inequality is based on rather delicate *probabilistic assumptions.* If one of these assumptions is not justified in the probability description of the EPR experiment, [8], [1]-[3], then there would be no Bell's inequality at all (or it should be modified [9]). Experiments of Aspect et al. [2] may be interpreted not only as arguments against the local realism, but also as experiments which examine the use of Bell's probabilistic assumptions. Of course, Bell's probabilistic assumptions are not just mathematical postulates. They must have some physical meaning. To find this meaning is an incredibly hard problem (at least on the present level of experiments with quantum systems). Therefore careful analysis of Bell's assumptions must be performed.

Such an analysis has already been performed [1]-[3] in the framework of the conventional probability theory (the measure-theoretical approach of A. N. Kolmogorov, 1933,[10]). In this approach probability is defined as a measure on the σ-field of events. The greatest advantage of this approach is its *abstractness.* All probability statements are obtained for abstract probability distributions. These statements can be used in the study of any physical model (without any specification). In the particular case of Bell's inequality this abstractness of Kolmogorov probability theory implies that Bell's inequality is obtained for abstract probability distribution on spaces of hidden variables (without any specification of internal structures of these spaces). The only restrictive condition on a probability distribution is the factorization condition (Bell-Clauser-Horne-Shimony locality condition [1]-[3] or outcome independence condition [3]):

$$\mathbf{p}(a=\alpha,\lambda=k,b=\beta)=\mathbf{p}(a=\alpha,\lambda=k)\ \mathbf{p}(b=\beta,\lambda=k)\ , \qquad (1)$$

where a and b are two settings of two measurement apparatuses, $\alpha,\beta=0,\pm1$, λ is the hidden variable which describes the complete state of two correlated particles. Typically the generality of Bell's inequality is considered as the remarkable physical result (no-go theorem), [1]-[3].[1]

[1] However, physical intuition must say that such a generality is rather suspicious from

The viewpoint that the notion of probability plays the large role in Bell's (and in EPR's) considerations is not so new, [4]-[7]. The main consequence of all these probabilistic analyses is that the EPR experiment could not be described by the unique Kolmogorov probability distribution (as it was assumed by J. Bell, [1]): (1) De Broglie, Lochak, Nelson, De Muynck, De Baere, Marten, Stekelenborg, [4], thermodynamical approach to Bell's problem, difference between hidden and observed probabilities; (2) Accardi [5], quantum probabilities, no Bayes' formula; (3) Pitowsky and Gudder [6], probability manifolds; (4) De Baere [6], fluctuating probabilities; (5) Fine and Rastal [6], no simultaneous probability distribution; (6) Muckenheim [6], negative probabilities; (7) Khrennikov [7], p-adic probabilities; fluctuating probabilities and modified Bell's inequality [9]. It would be important for our further considerations to remark that Pitowsky and Gudder have analysed the role of the notion of an event in Bell's considerations (see also Shimony [3]); Accardi have analysed the role of Bayes' axiom (the definition of a conditional probability in Koilmogorov's probability model).

In the present paper we study Bell's probabilistic assumptions (their mathematical and physical origins) on the basis of the frequency probability theory of R. von Mises, 1919, see [11] for the advanced theory. The main advantage of Mises' approach is that there we use the primary random characteristics of physical phenomena, namely **relative frequencies.** This theory (in the opposite to Kolmogorov's one) is concrete. It describes not probability distributions by , but some 'underground random world', which may (or may not) produce probability distributions. We perform analysis of existence of collectives (random sequences) and their properties (under the additional assumption that they exist) in the EPR experiments. It would be impossible to perform such an analysis in the conventional probabilistic framework in that abstract probability distributions are considered as they are given by God. In the latter case it is practically impossible to find physical sources of existence or nonexistence of these probabilities (as well as justify some of their properties).

Our frequency analysis demonstrated that nonexistence of some collectives (and, as consequence, some probability distributions) can be one of possible sources of the violation of Bell's probabilistic 'postulates'. Of course,

the physical viewpoint. It would be more natural for a purely mathematical theorem which must hold true for all models described by one fixed system of axioms (in Bell's case by Kolmogorov's axiomatics).

such nonexistence of collectives (the absence of the statistical stabilization of relative frequencies) can be interpreted as the impossibility of the realistic description. However, this is right only as long as we use the EPR approach to the notion of an *element of reality,* [8]. In fact, EPR's notion of an element of reality is strongly based on Kolmogorov's probability (one) of events. First of all we remark that even EPR noticed that they could not give a 'definition' of an element of reality. EPR proposed some sufficient condition. Thus in principle we could introduce other elements of reality which in general do not satisfy EPR's sufficient condition. We do not try to do this in the present paper.[2] In any case our frequency analysis demonstrated that the EPR notion of an element of reality is closely connected with one special probability model, namely Kolmogorov's one.

Further frequency analysis demonstrated that even if we postulate that all collectives exist [3], there arises a new problem (which did not present in Kolmogorov's framework). This is the question on the possibility of *combining* [11] collectives corresponding to different settings of measurement devices. The operation of combining of two different collectives (random sequences) $x = (x_1, x_2, ..., x_N, ..)$ and $y = (y_1, y_2, ..., y_N, ..)$ (to obtain a new collective $z = (z_j), z_j = \{x_j, y_j\}$) is a rather delicate operation, see von Mises [11] (see also [9]). At the moment there are no physical reasons to suppose that EPR-like collectives are combinable. We remark that there is nothing especially "quantum" in the existence of noncombinable collectives, see [11], [9]. If we even suppose that collectives are combinable, then, to derive Bell's inequality, we must also assume that these collectives are *independent.* The latter assumption is even more doubtful, see our analysis. Moreover, it must be underlined that in the frequency framework the violation of condition (10) need not be interpreted as the evidence of nonlocality. The nonlocal viewpoint [1]-[3] is merely a consequence of the conventional viewpoint to conditional probabilities, namely as probabilities of events. Here dependence means a dependence of events. On the other hand, in the frequency framework dependence means dependence of collectives. There is no nonlocal influence at

[2]One of such models of 'fluctuating reality' was proposed in [7] on the basis of so called p-adic probability. The notion of fluctuation of frequencies depends crucially on the choice of a topology. There exist random phenomena such that frequencies fluctuate in the real metric (no 'stable probability distributions'), but stabilize in a p-adic metric. This is a kind of reality which is described by other numbers (or labels) than real numbers.

[3]So fluctuations of hidden variables for quantum particles and apparatuses are not completely chaotic. They have the property of the statistical stabilization.

all (in any case we need not apply to such an influence). We also discuss the problem of transmission of information with the aid of EPR pairs.

Of course, our frequency analysis could not be considered as an argument against nonlocality. The quantum reality may be nonlocal. However, we demonstrate that Bell's inequality is based on so doubtful probabilistic considerations that it could not be considered as a serious argument against the local realism.

As the frequency probability theory is now days practically forgotten, we must to present an introduction to this approach (see section 1).

2 Frequency probability theory

2.1. History. The frequency probability theory was developed by R. von Mises in 1919 (see [11], [9] for the details). In fact, the basis of the frequency approach was provided in the work of J. Venn, 1866, see [9]. The frequency theory was used as the motivation of Kolmogorov's axiomatic, 1933, of the conventional probability theory (see remarks in [10]). The main advantage of the conventional theory is its *abstractness.* Here we work with abstract probability distributions which are not directly related to the concrete physical model. Thus results of the conventional probability theory can be used without any modification in any physical models. However, this advantage may become in some circumstances a disadvantage, because the abstractness of the formalism does not give the possibility to analyse the origin (and even the existence) of probability distributions. On the other hand, the frequency theory of probability is concrete. Here to introduce a probability distribution, we must be sure that there exists a collective (random sequence) which produces this probability distribution. The collective is more primary object than a probability distribution. The collective has more direct connection with a physical phenomenon. However, in the frequency approach we cannot obtain results which are valid for 'all probability distributions'. The probability distribution without a collective is nothing. Typically such a concreteness is considered as the large disadvantage of the frequency approach (comparing with the conventional measure theoretical approach). Of course, it is more attractive to prove some probabilistic statement ones and then to apply it to numerous physical models. This was one of the reasons to eliminate the frequency approach from applications in the favour of the measure-theoretical approach. [4]

In the present paper we demonstrate that the frequency analysis of probabilistic

[4] Another reason was the problem of the rigorous mathematical definition of a collective, random sequence, see, for example, [9].

assumptions for the derivation of Bell's inequality can give some new sights to this problem. These sights would be impossible to obtain in the conventional abstract framework. Analysis of collectives can give more than analysis of abstract probability distributions.

2.2. Collective. Let $\mathcal{E}$ be an ensemble of physical systems. We take elements of $\mathcal{E}$ and form a sequence $\pi = (\pi_1, \pi_2, ..., \pi_N, ...)$. Suppose that elements of $\mathcal{E}$ have some properties. [5] Suppose that these properties can be described by natural numbers, $L = \{1, 2, ..., m\}$ (the set of 'labels'). In principle we can consider continuous label sets, see [11]. Thus, for each $\pi_j \in \pi$, we have a number $\alpha_j \in L$. So π induces a sequence

$$x = (\alpha_1, \alpha_2, ..., \alpha_N, ...), \quad \alpha_j \in L. \tag{2}$$

For each fixed $\alpha \in L$, we have the relative frequency $\nu_N(\alpha) = n_N(\alpha)/N$ of the appearance of α in $(\alpha_1, \alpha_2, ..., \alpha_N)$.

R. von Mises said that x satisfies to the principle of the *statistical stabilization* of relative frequencies, if, for each fixed $\alpha \in L$, $|\nu_N(\alpha) - \nu_M(\alpha)| \to 0, N, M \to \infty$. The corresponding limit

$$\mathbf{p}(\alpha) = \lim_{N\to\infty} \nu_N(\alpha) \tag{3}$$

is said to be a probability. This probability can be extended to the field of all subsets of L :

$$\mathbf{p}(B) = \lim_{N\to\infty} \nu_N(\alpha \in B) = \lim_{N\to\infty} \sum_{\alpha\in B} \nu_N(\alpha) = \sum_{\alpha\in B} \mathbf{p}(\alpha) \ , B \subset L \tag{4}$$

(the situation becomes sufficiently complex for an infinite L, see Tornier [11]). We remark that $\mathbf{p}(L) = 1$.

R. von Mises said that x satisfies the principle of *randomness* if limits (3) are invariant with respect to choices of some subsequences in x. These choices of subsequences, so called place selections, have some properties, see [11] or [9] (which are unimportant for our investigation). [6] In principle the reader may forget about the principle of randomness and consider only the principle of the statistical stabilization. It seems that only this principle is

[5] It is not important in general either these properties are objective (properties of an object) or 'created' in the process of observation by an observer, see [3].

[6] The class of place selections was not defined precisely by R. von Mises. This induced numerous discussions. However, the problem can be solved (at least partially) by the consideration of countable classes of place selections, Wald theorem, [11] or [9], p.43.

important (at least at the moment) in physics in that we study behaviour of frequencies.

Sequence (11) which satisfies to two von Mises' principles is said to be a *collective*; $\mathbf{p}$ is said to be a *probability distribution* of the collective x. We will often use the symbols $\mathbf{p}(B;x)$ (and $\nu_N(B;x), n_N(B;x)), B \subset L$, to indicate the dependence on the concrete collective x.

The frequency probability formalism is not a calculus of probabilities. It is a *calculus of collectives.* Thus instead of operations for probabilities (as it is in the conventional probability theory), we define operations for collectives.

2.3. Operation of combining of collectives. This operation will play the crucial role in our analysis of probabilistic foundations of Bell's arguments. Let $x = (x_j)$ and $y = (y_j)$ be two collectives with label sets L_x and L_y, respectively. We define a new sequence $z = (z_j),\ z_j = \{x_j, y_j\}$ (in general z is not a collective). Let $a \in L_x$ and $b \in L_y$. Among the first N elements of z there are $n_N(a;z)$ elements with the first component equal to a. As $n_N(a;z) = n_N(a;x)$ is a number of $x_j = a$ among the first N elements of x, we obtain that $\lim_{N\to\infty} \frac{n_N(a;z)}{N} = \mathbf{p}(a;x)$. Among these $n_N(a;z)$ elements, there are a number, say $n_N(b/a;z)$ whose second component is equal to b. The frequency $\nu_N(a,b;z)$ of elements of the sequence z labeled (a,b) will then be

$$\frac{n_N(b/a;z)}{N} = \frac{n_N(b/a;z)}{n_N(a;z)} \ \frac{n_N(a;z)}{N} \ .$$

We set $\nu_N(b/a;z) = \frac{n_N(b/a;z)}{n_N(a;z)}$. Let us assume that, for each $a \in L_x$, the subsequence $y(a)$ of y which is obtained by choosing y_j such that $x_j = a$ is an collective. Then, for each $a \in L_x$, $b \in L_y$, there exists

$$\mathbf{p}(b/a;z) = \lim_{N\to\infty} \nu_N(b/a;z) = \lim_{N\to\infty} \nu_N(b;y(a)) = \mathbf{p}(b;y(a)). \tag{5}$$

We have $\sum_{b\in L_2} \mathbf{p}(b/a;z) = 1$. The existence of $\mathbf{p}(b/a;z)$ implies the existence of $\mathbf{p}(a,b;z) = \lim_{N\to\infty} \nu_N(a,b;z)$. Moreover, we have

$$\mathbf{p}(a,b;z) = \mathbf{p}(a;x)\ \mathbf{p}(b/a;z) \tag{6}$$

and $\mathbf{p}(b/a;z) = \mathbf{p}(a,b;z)/\mathbf{p}(a;x)$, if $\mathbf{p}(a;x) \neq 0$. We have

$$\sum_{a\in L_a} \sum_{b\in L_2} \mathbf{p}(a,b;z) = 1.$$

Thus in this case the sequence z is an collective and the probability distribution $\mathbf{p}(a,b;z)$ well defined. The collective y is said to be *combinable* with the collective x. The relation of combining is a symmetric relation on the set of pairs of collectives with strictly positive probability distributions ($\mathbf{p} > 0$).

2.4. Independent collectives. Let x and y be collectives. Suppose that they are combinable. The y is said to be independent from x if all collectives $y(a)$, $a \in L_x$, have the same probability distribution which coincides with the probability distribution $\mathbf{p}(b; y)$ of y. This implies that

$$\mathbf{p}(b/a; z) = \lim_{N \to \infty} \nu_N(b/a; z) = \lim_{N \to \infty} \nu_N(b; y(a)) = \mathbf{p}(b; y) \ .$$

Here the conditional probability $\mathbf{p}(b/a; z)$ does not depend on a. Hence

$$\mathbf{p}(a, b; z) = \mathbf{p}(a; x) \ \mathbf{p}(b; y), \ a \in L_x, \ b \in L_y.$$

From the physical viewpoint the notion of independent collectives is more natural than the notion of independent events in the conventional probability theory. In latter the relation $\mathbf{p}(a, b) = \mathbf{p}(a)\mathbf{p}(b)$ can hold just occasionally (as the result of a game with numbers, see [11] or [9], p.53).

3 Collectives associated with hidden variables description of the EPR experiment

3.1. Hidden variable description. We consider the standard EPR framework. Settings of two different measurement apparatuses (for particles 1 and 2, respectively) will be denoted, respectively, by $a, a', a''...$ and $b, b', b'', ...$; it is supposed that $a, b, a', b', ... = 0, \pm 1$;[7] hidden variables are denoted by λ; the set of hidden variables is finite $\Lambda = \{1, 2, ..., M\}$. Internal microstates of measurement apparatuses are described by variables $\omega_a, \omega_b, ...$ (see Bell [1]); sets of these microstates are also finite: $\Omega_a = \Omega_b = ... = \{1, ..., T\}$.

A sequence of pairs of particles $\pi = \{\pi_j = (\pi_j^1, \pi_j^2), j = 1, 2, ...\}$ is prepared for the same quantum state ψ.[8] Let $\lambda_j \in \Lambda, j = 1, 2, ...$ be the value of the hidden variable for the jth pair.

[7] We consider the value $\alpha = 0$ to include into our study models (with CHSH-inequality) which take into account the efficiency of detectors.

[8] If we follow to the orthodox Copenhagen interpretation, then we suppose that ψ gives the complete description of each quantum system π_j (a pair of particles) under the consideration. If we follow to the statistical interpretation of quantum mechanics, see, for example, [12], we suppose that ψ describes (some?) statistical properties of the ensemble π of quantum systems.

3.2. Existence of collectives . For settings a and b of measurement apparatuses we consider sequences of pairs

$$x_{\omega_a,\lambda} = \{(\omega_{a1}, \lambda_1), ..., (\omega_{aN}, \lambda_N), ...\} ,$$

$$x_{\omega_b,\lambda} = \{(\omega_{b1}, \lambda_1), ..., (\omega_{bN}, \lambda_N), ...\} ,$$

where ω_{aj} and ω_{bj} are internal states of apparatuses at the moments $j = 1, 2, ...$ of interactions with particles π_j^1 and π_j^2, respectively.

The first question is the following:

$\mathbf{Q}_1$: *Are these sequences collectives?*

There are no strong physical reasons for the positive answer. Both a preparation device (which produces particles) and measurement devices $a, b, ..$ are complex systems. There are no reasons to suppose that their microfluctuation must produce the statistical stabilization of frequencies: $\nu_N(\omega_a = s, \lambda = k)$ $(\nu_N(\omega_b = q, \lambda = k), ...)$ for fixed $s \in \Omega_a, k \in \Lambda$ $(q \in \Omega_b, ...)$. The reader may think that such a fluctuation of frequencies (the absence of the probability distribution $\mathbf{p}(\omega_a = s, \lambda = k)$ $(\mathbf{p}(\omega_b = q, \lambda = k), ...)$ must contradict to the statistical stabilization for the results of observations $a = \alpha, b = \beta, ...$, where $\alpha, \beta = 0, \pm 1$. Let as denote by $\Sigma_a(\alpha)$ the set of pairs (ω_a, λ) which produce the value $a = \alpha$ for the apparatus a (with similar notations for other apparatuses and their settings).[9]

Then

$$\mathbf{p}(a = \alpha) = \lim_{N \to \infty} \sum_{(s,k) \in \Sigma_a(\alpha)} \nu_N(\omega_a = s, \lambda = k). \tag{7}$$

Such a limit of the average with respect to the set $\Sigma_a(\alpha)$ can exist despite the fluctuations of frequencies $\nu_N(\omega_a = s, \lambda = k)$ for fixed s and k (see appendix 1).

Remark 3.1. (Physical meaning of nonexistence of collectives). Of course, in the real physical experiment we cannot generate an infinite sequence of trials. Thus the notion of a collective is just a mathematical idealization. Nevertheless, the absence of the statistical stabilization has the natural physical interpretation: if we consider two different (sufficiently long) runs, R_1 and R_2, of an experiment, then the frequencies $\nu_N(\alpha; R_1)$ and

[9] This is a contextualistic model with hidden variables, see De Muynck el al. [4]: the value of a physical observable A depends not only on the value of hidden variable λ for a quantum system, but also on the value of hidden variable ω_a for a measurement apparatus a.

$\nu_N(\alpha; R_2)$ (where α is a label) do not coincide even approximately. There exists a constant $\epsilon > 0$ (a measure of fluctuations) such that

$$\epsilon = \max_{R_1, R_2} \max_{\alpha} |\nu_N(\alpha; R_1) - \nu_N(\alpha; R_2| > 0.$$

In quantum experiments it is assumed that all physical systems are prepared in the same quantum state ψ. Thus $\epsilon = \epsilon_\psi$ can be considered as probability invariant of the quantum state ψ (see [9]). Of course, such a nonreproducibility for hidden variables for systems which are described by the fixed quantum state is closely related to noncompleteness of quantum mechanics, see section 4.3 (and may be to the efficiency of detectors, see section 5.2). In [9] we obtained a modification of Bell's inequality in that it is taken into account the contribution of distribution fluctuations:

$$| < a, b > - < c, b > | \leq (1 + 2\epsilon) - < a, c > \ . \tag{8}$$

In the same way we can obtain a modification of CHSH-inequality:

$$-2 - 4\epsilon_\psi \ \leq < a', b' > + < a', b'' > + < a'', b' > - < a'', b'' > \leq \ 2 + 4\epsilon \ . \tag{9}$$

Thus fluctuations of statistical distributions of microparameters can induce essential macroeffects.

The question on the existence of collectives $x_{\omega_a, \lambda}$ $(x_{\omega_b, \lambda}, ...)$ can be reduced to the question on combining of collectives corresponding, respectively, to states of particles and apparatuses.

Let us consider sequences

$$x_\lambda = (\lambda_1, ..., \lambda_N, ...) \ \ , x_{\omega_a} = (\omega_{a1}, ..., \omega_{aN}, ...) \ .$$

The first question is again $\mathbf{Q}_1$. We again have to recognize that at the moment we cannot give the definite answer to this question. In particular, for x_λ, this is nothing than the problem of the *reproducibility* of hidden variables, De Baere [6] (see also [9]). If x_λ is not a collective, then in different runs of experiments we would obtain different 'probability distributions' of hidden variables.

3.3. On combining of collectives. Suppose that the law of the statistical stabilization is not only a law of macrophysics (a consequence of averages with respect to huge ensembles of microstates), but also a law of microphysics. Thus x_λ and x_{ω_a} are collectives. The next question is the following:

$\mathbf{Q}_2$: *Are these collectives combinable?*

They are combinable if frequencies $\nu_N(\omega_a = s, \lambda = k)$ (to find the fixed value $\lambda = k$ for the quantum particle and the fixed value $\omega_a = s$ for the internal state of the apparatus) stabilize. It is not clear why we have to have such a statistical consistency.

3.4. On independence of collectives for the same apparatus. Suppose that x_λ and x_{ω_a} are combinable collectives. Thus $x_{\omega_a,\lambda}$ is also a collective and the probability distribution $\mathbf{p}(\omega_a = s, \lambda = k)$ is well defined. In fact, to proceed the derivation of CHSH inequality (see Bell's proof [1]) we have to use the factorization condition

$$\mathbf{p}(\omega_a = s, \lambda = k) = \mathbf{p}(\omega_a = s)\ \mathbf{p}(\lambda = k)\ .$$

Therefore combinable collectives x_λ and x_{ω_a} have to be independent. However, if in the process of the interaction a quantum particle has some influence to the micro state of a measurement apparatus, then the assumption on independence is not justified. Such an influence can be negligibly small for each individual particle, but the integral effect can imply macro consequences (via (7)). In particular, dependence is natural for 'thermodynamic models' [4] (see also appendix 2).

3.5. Combinable or uncombinable? Let us assume that $x_{\omega_a,\lambda}$ and $x_{\omega_b,\lambda}$ are collectives. Thus frequency probabilities $\mathbf{p}(\omega_a = s, \lambda = k), \mathbf{p}(\omega_b = q, \lambda = k)$ are well defined. We again have to ask question $\mathbf{Q}_2$. Let us write the condition of combining:

$$\nu_N(\omega_a = s, \lambda = k, \omega_b = q) = \frac{n_N(\omega_a = s, \lambda = k, \omega_b = q)}{N} =$$

$$\frac{n_N(\omega_a = s, \lambda = k, \omega_b = q)}{n_N(\omega_a = s, \lambda = k)}\ \frac{n_N(\omega_a = s, \lambda = k)}{N} =$$

$$\nu_N(\omega_b = q, \lambda = k/\omega_a = s, \lambda = k)\ \nu_N(\omega_a = s, \lambda = k) \to$$

$$\mathbf{p}(\omega_b = q, \lambda = k/\omega_a = s, \lambda = k)\ \mathbf{p}(\omega_a = s, \lambda = k)\ ,\ N \to \infty.$$

Under the assumption that $x_{\omega_a,\lambda}$ is a collective, we have that $\frac{n_N(\omega_b=q,\lambda=k/\omega_a=s,\lambda=k)}{n_N(\omega_a=s,\lambda=k)}$ must have the definite limit. What are physical reasons for such a statistical stabilization?

The case in that probabilities $\mathbf{p}(\omega_a = s, \lambda = k), \mathbf{p}(\omega_b = q, \lambda = k)$ are well defined, but $\mathbf{p}(\omega_a = s, \lambda = k, \omega_b = q)$ oscillate can be better understand on the basis of the following example.

Example 3.1. (Uncombinable collectives). Let A be the set of even numbers. Take any subset $C \subset A$ such that

$$\frac{1}{N}|C \cap \{1, 2, \cdots, N\}|$$

is oscillating. Here the symbol $|D|$ denotes the number of elements in the set D. There happen two cases: $C \cap \{2n\} = \{2n\}$ or $= \emptyset$. Set

$$B = C \cup \{2n - 1 : C \cap \{2n\} = \emptyset\}.$$

Suppose that, in the sequence $x_{\omega_a,\lambda}$, we have $\omega_a = s$ and $\lambda = k$ for trails $j \in A$, and, in the sequence $x_{\omega_b,\lambda}$, we have $\omega_b = q$ and $\lambda = k$ for trails $j \in B$. Both frequency probabilities $\mathbf{p}(\omega_a = s, \lambda = k)$ and $\mathbf{p}(\omega_b = q, \lambda = k)$ are well defined and equal to $1/2$. However, the probability $\mathbf{p}(\omega_a = s, \lambda = k, \omega_b = q)$ is not defined.

3.6. On independence of collectives for different apparatuses. Let us assume that $x_{\omega_a,\lambda}$ and $x_{\omega_b,\lambda}$ are combinable collectives. Thus the probability distribution $\mathbf{p}(\omega_a = s, \lambda = k, \omega_b = q)$ is well defined. To proceed the derivation of CHSH inequality, we have to have factorization condition

$$\mathbf{p}(\omega_a = s, \lambda = k, \omega_b = q) = \mathbf{p}(\omega_a = s, \lambda = k)\ \mathbf{p}(\omega_b = q, \lambda = k)\ . \qquad (10)$$

Therefore it must be supposed that collectives $x_{\omega_a,\lambda}$ and $x_{\omega_b,\lambda}$ are independent. However, they both contain the same parameter λ. This is a kind of constraint. Hence, from the general point of view, they have to be dependent! There must be special reasons by that in the EPR-experiment these collectives are independent despite the λ-constraint. I do not see such reasons. If we assume that the result of a measurement is (more or less) determined by λ, then it seems that sequences $x_{a,\lambda}$ and $x_{b,\lambda}$ must be dependent. Thus factorization condition (10) in general must be violated.

3.7. Statistical independence. Let us go outside the domain of von Mises' frequency probability theory and ask the question: *Are uncombinable collectives dependent or independent?* Of course, here we can use only some heuristic reasons. It seems that we may consider uncombinable collectives as *statistically independent:* a special choice of trials in one collective could not provide some statistical information about the corresponding trials in another collective.

3.8. Averages with respect to microstates of apparatuses. Typically Bell's framework for the EPR experiment is described without the introduction of hidden variables for apparatuses $\omega_a, \omega_b, \ldots$ In such a case only probabilities $\mathbf{p}(a = \alpha, \lambda = k)$ are used. Thus in the frequency analysis we must consider sequences

$$x_{a,\lambda} = \{(a_1, \lambda_1), ..., (a_N, \lambda_N), ...\} \ , \tag{11}$$

$$x_{b,\lambda} = \{(b_1, \lambda_1), ..., (b_N, \lambda_N), ...\} \ , \tag{12}$$

where a_j and b_j are the jth results for a and b. The corresponding frequency analysis will be presented in section 3.9. We now discuss the possibility of the transition from probabilities $\mathbf{p}(\omega_a = s, \lambda = k)$ to probabilities $\mathbf{p}(a = \alpha, \lambda = k)$.

Such a transition is not so innocent. It was evident even for authors using Kolmogorov's measure theoretical viewpoint to probability (see Shimony [3] and Shimony, Clauser, Horne [2]). We want to present a few remarks on the use of hidden variables (microstates) $\omega_a, \omega_b, ..$ for apparatuses as well as on the critique of this use. First of all we remark that from the physical viewpoint there are no reasons to eliminate $\omega_a, \omega_b, ..$ from Bell's framework. If we want to develop the consistent theory of hidden variables, then we have to take into account hidden variables of particles as well as hidden variables of apparatuses. This is in the complete agreement with the general viewpoint on quantum measurements (an extension of Bohr's views to hidden variable description), see, for example, De Muynck et al. [4] for the detailed analysis. Such a viewpoint was strongly supported by J. Bell [1]. The critique of the use of hidden variables $\omega_a, \omega_b, ..$ is merely the critique of the possibility to describe such a situation with the aid of Kolmogorov measures. We support this critique. We improve critical arguments by our frequency analysis.

Let $\alpha = 0, \pm 1, k \in \Lambda$. Set

$$\sigma_a(\alpha; k) = \{s \in \Omega_a : a(s, k) = \alpha\},$$

where $a = a(\omega_a, \lambda)$ is the result of a measurement for the state ω_a of a and the state λ of a quantum particle. Suppose that $x_{\omega_a,\lambda}$ is a collective. The frequency probabilities $\mathbf{p}(\omega_a = s, \lambda = k)$ are well defined. We have

$$\mathbf{p}(a = \alpha, \lambda = k; x_{a,\lambda}) = \lim_{N\to\infty} \sum_{s\in\sigma_a(\alpha;k)} \nu_N(\omega_a = s, \lambda = k; x_{\omega_a,\lambda}).$$

If the set Ω_a of microstates of a is finite, then we can make the change:

$$\lim_{N\to\infty} \sum_s = \sum_s \lim_{N\to\infty} \tag{13}$$

and obtain that the probability $\mathbf{p}(a=\alpha, \lambda=k; x_{a,\lambda})$ is well defined (so $x_{a,\lambda}$ is a collective)

$$\mathbf{p}(a=\alpha, \lambda=k; x_{a,\lambda}) = \sum_{s\in\sigma_a(\alpha k)} \mathbf{p}(\omega_a = s, \lambda=k; x_{\omega_a,\lambda}).$$

Suppose that Ω_a is infinite. Then, in general, we cannot perform (13). Thus the assumption that $x_{\omega_a,\lambda}$ is a collective need not imply that $x_{a,\lambda}$ is a collective.

Remark 3.2. (Physical meaning of infinite sets of microstates of apparatuses) The mathematical assumption that $\Omega_a, \Omega_b, ...$ are infinite sets has the following physical interpretation. The number of possible microstates of the apparatus a which produce the same value $a=\alpha$ is very large. In different runs, R_1, R_2, of the experiment we can obtain essentially different parts of the set $\sigma_a(\alpha; k)$, namely $\sigma_a(\alpha; k; R_1), \sigma_a(\alpha; k; R_2)$. It may even be that $\sigma_a(\alpha; k; R_1) \cap \sigma_a(\alpha; k; R_2) = \emptyset$. Thus 'probabilities' $\mathbf{p}(a=\alpha, \lambda=k; R_j) = \sum_{s\in\sigma_a(\alpha;k;R_j)} \nu_N(\omega_a = s, \lambda=k; R_j), j=1,2,$ may strongly differ.

3.9. Description without hidden variables for the apparatus. In principle, we may start directly with the consideration of sequences $x_{a,\lambda}$, (11), and $x_{b,\lambda}$, (12), and repeat our frequency analysis. In fact, here we have similar problems, in particular, questions $\mathbf{Q}_1$ (existence) and $\mathbf{Q}_2$ (combining) and the problem of independence of collectives. Independence of collectives is equivalent to the factorization of the simultaneous probability distribution:

$$\mathbf{p}(a=\alpha, \lambda=k, b=\beta; x_{a,\lambda,b}) = \mathbf{p}(a=\alpha, \lambda=k; x_{a,\lambda})\; \mathbf{p}(b=\beta, \lambda=k; x_{b,\lambda})\;. \tag{14}$$

4 Physical consequences

4.1. Nonlocality or dependence. Suppose that there is no chaos on the micro level. Thus the statistical behaviour of hidden variables and their combinations with macro observables is always characterized by the statistical stabilization. Suppose also that huge systems of internal parameters of two measurement devices and hidden variables of quantum particles are in the statistical consistency, so collectives $x_{a,\lambda}$ and $x_{b,\lambda}$ are combinable. Under such assumptions the only root of the absence of the Bell inequality may be the dependence of collectives $x_{a,\lambda}$ and $x_{b,\lambda}$ (which imply the violation of factorization conditions (14)).

In the investigations based on the conventional probability theory [1]-[3] condition (1) is typically considered as the condition of *nonlocality.*[10]

This implies numerous inferences of the relation between the quantum mechanics and general relativity. In our frequency analysis there is no trace of nonlocality in the dependence of collectives $x_{a,\lambda}$ and $x_{b,\lambda}$. They are dependent due to the present of the same parameter λ in both collectives.

I think that one of the reasons for different interpretations the violation of factorization condition (1) is the difference in the meaning of a conditional probability in the conventional and frequency theories of probabilities.

In the conventional approach a conditional probability is probability for events, namely $\mathbf{p}(B/A)$ has the meaning of the probability of the event B under the condition of the event A. Here $\mathbf{p}(B/A) \neq \mathbf{p}(B)$ implies that the event B depend on the event A. Specially in the EPR framework the violation of (10) implies that the event $B = \{$ obtain the value $b = \beta$ for a particle 2 with $\lambda = k\}$ depends on the event $A = \{$ obtain the value $a = \alpha$ for a particle 1 with $\lambda = k\}$. In principle such a dependence of events may be interpreted as an evidence of nonlocalty.

In the frequency framework a conditional dependence (or independence) is related not to events, but to collectives. Thus condition (14) only implies that collectives are dependent. However, there is nothing surprising in such a dependence (since particles 1 and 2 are correlated).

4.2. Transmission of information with the aid of dependent collectives. There were numerous discussions on the possibility to use 'nonlocality condition'

$$\mathbf{p}(a = \alpha, \lambda = k, b = \beta) \neq \mathbf{p}(a = \alpha, \lambda = k)\, \mathbf{p}(b = \beta, \lambda = k) \tag{15}$$

for the transmission of information, see, for example, [3]. Typically such a transmission of information was connected with 'essentially quantum' properties (so called entanglement). However, the standard scheme can be applied to transfer information with the aid of any two dependent collectives. Let $u = (u_j)$ and $v = (v_j)$ be dependent collectives and let here $\alpha, \beta = \pm 1$. As they are combinable, conditional probabilities

$$\mathbf{p}(v = \beta/u = \alpha) = \lim_{N\to\infty} \nu_N(v = \beta; v(\alpha)), \alpha, \beta = \pm 1,$$

[10]More neutral terms are used by some authors. For example, A. Shimony called it 'outcome independence', [3]. De Muynck [4] used the term 'conditional statistical independence.' However, he still used the conventional measure-theoretical approach to probability. There is a crucial difference in the viewpoints on independence in the conventional and frequency theories, see our further considerations.

are well defined. Here, as usual, $v(\alpha)$ is a collective obtained from v by the choice of subsequence v_{j_k} such that $u_{j_k} = \alpha$. As collectives are dependent, we have, for example,

$$\mathbf{p}_1 = \mathbf{p}(v = 1/u = +1) \neq \mathbf{p}_2 = \mathbf{p}(v = 1/u = -1).$$

Then all is the same as in all 'quantum stories'. Bob prepares a statistical ensemble of pairs which components are described by collectives u and v respectively. He chooses subcollective $v(+1)$ and sends it to Alice. If Alice knows the relation between probabilities, she can easily rediscover the bit of information.

4.3. Fluctuating ensemble distributions and the completeness of quantum mechanics. The main aim of the paper [8] of A. Einstein, B. Podolsky and N. Rosen was to demonstrate that 'the quantum mechanical description of reality given by the wave function is not complete.' The EPR considerations (as well as their further modifications, see, for example books [3]) imply that either the quantum mechanical description of reality is not complete or the local realism must be eliminated from quantum formalism. There is quite general opinion that Bell's considerations [1]-[3] transformed the EPR-alternative into a new (Bell's) alternative: either nonlocality or nondeterminism. Our considerations demonstrated that such a viewpoint is not completely justified. It seems that, despite Bell's inequality and its generalizations, we still have the EPR-alternative. In fact, J. Bell supposed that the quantum mechanical description of reality is complete: a quantum state ψ determines uniquely the probability distribution of hidden variables.

It may seem that the ensemble dependence of distributions of hidden variables imply that these hidden variables cannot be considered as objective properties (properties of an object, see [3]). However, it is not so. They are really properties of objects. There is nothing special that the statistical distribution of these properties may vary from ensemble to ensemble (such a behaviour is standard in classical physics, economy, biology). It may seem that the statistical fluctuations of hidden variables contradict to the statistical stability of physical observables. However, it is not so. We illustrate this by a simple example, see appendix 1.

At the moment we cannot determine the definite source of ensemble fluctuations. This problem may have a trivial (from the ontological viewpoint) solution: ensemble fluctuations are a consequence of a low efficiency of detectors, see section 5.2. However, it may be that it is the fundamental property of quantum reality.

5 Links to some measure-theoretical results

In this section we present connections with some well known results on Bell's inequality which were obtained on the basis of Kolmogorov probability model.

5.1. Existence of the simultaneous probability distribution for $n \geq 3$ settings. It was proved by Fine and Rastall [6] that Bell's inequality is equivalent to the existence of the simultaneous probability distribution for three different settings a, a', b of measurement apparatuses (for GHSH inequality we have to consider four different settings). As usual in this paper symbols a, a' and b are used, respectively, for settings of the measurement devices for the first particle and second particle. We analyse the FR-framework from the frequency viewpoint.

As it has been mentioned, in the frequency theory we could not consider a probability distribution without relation to some collective. However, the object which is called a 'probability distribution' in the FR-framework in general has no relation to a collective. So such an object has no probabilistic (and, consequently, physical) meaning from von Mises' viewpoint.[11] The FR-condition is just purely mathematical constraint.

If we accept the use of counterfactuals, then we can continue the frequency analysis of the FR-arguments. Beside of collectives $x_{a,\lambda} = \{(a_j, \lambda_j), j = 1, 2, ...\}, x_{b,\lambda} = \{(b_j, \lambda_j), j = 1, 2, ...\}$, we can consider 'gedanken kollektiv' $x_{a',\lambda} = \{(a'_j, \lambda_j), j = 1, 2, ...\}$. Suppose that three collectives are combinable. There exists the simultaneous probability distribution $(\alpha, \beta, \gamma = 0, \pm 1)$:

$$\mathbf{p}(a = \alpha, b = \beta, a' = \gamma, \lambda = k) = \lim_{N \to \infty} \frac{1}{N} \nu_N(a = \alpha, b = \beta, a' = \gamma, \lambda = k; x_{a,b,a',\lambda}) \ .$$

The average with respect to λ (if such a procedure is justified) gives the simultaneous probability distribution:

$$\mathbf{p}(a = \alpha, b = \beta, a' = \gamma) = \lim_{N \to \infty} \frac{1}{N} \nu_N(a = \alpha, b = \beta, a' = \gamma; x_{aba'}) \ . \qquad (16)$$

In this case we can apply the FR-theory and obtain Bell's inequality without the assumption that collectives $x_{a,\lambda}$ and $x_{b,\lambda}$ are independent (i.e., without factorization condition (1)). It is right.

We now suppose that three collectives $x_{a,\lambda}, x_{b,\lambda}, x_{a',\lambda}$ are not combinable. Thus limit (16) does not exist. There is no simultaneous probability distribution $\mathbf{p}(a = \alpha, b = \beta, a' = \gamma)$. However, it can be that there exists real

[11]Eberhard [2] rightly pointed out that Fine's statements contain rather unclear words on simultaneous probability distribution: "well defined."

numbers $\mathbf{p}_{\alpha\beta\gamma} \geq 0, \sum \mathbf{p}_{\alpha\beta\gamma} = 1$ such that $\mathbf{p}(a = \alpha, b = \beta; x_{ab}) = \sum_\gamma \mathbf{p}_{\alpha\beta\gamma}$. By FR-result we have Bell's inequality.

This identification of mathematical FR-constants with physical probabilities is the root of the deep misunderstanding of the role of the FR-result. This result is often interpreted as the demonstration that BCHS locality condition is not directly related to Bell's inequality. The violation of Bell's inequality is connected with the fact that observables a and a' are incompatible: this implies the absence of the simultaneous probability distribution even for two observables a and a'. It is the mistake. Such an inference might be done if we could prove that Bell's inequality must imply the existence of frequency probability distribution (16). However, it is not so.

Thus we have shown that BCHS locality condition (1) (which is regarded as the collective independence condition in von Mises' approach) cannot be eliminated from Bell's framework by the FR-result. Yes, we know that there is no simultaneous probability distribution (16) in the EPR experiment (as the observables a and a' are incompatible). However, even in such a situation condition (1) implies Bell's inequality.[12]

5.2. Frequency viewpoint to the efficiency of detectors. There are numerous results which demonstrate that the problem of the efficiency of detectors play the important role in Bell's framework, see, for example, [2], [3]. Our frequency analysis demonstrated that low efficiency of detectors may be a root of some 'pathologies'.

(a) Sequence $x_{a,\lambda}$ may be not a collective (so $\mathbf{p}(a = \alpha, \lambda = k), \alpha = 0, \pm 1, k \in \Lambda$ may fluctuate). In different runs, R_1, R_2, the a-apparatus records ensembles, $\mathcal{E}_1, \mathcal{E}_2$, of particles with essentially different distributions of hidden variables. Therefore frequencies $\nu_N(a \neq 0, \lambda = k; R_1)$ and $\nu_N(a \neq 0, \lambda = k; R_2)$ may differ (even for large samples): $\max_{k \in \Lambda} |\nu_N(a \neq 0, \lambda = k; R_1) - \nu_N(a \neq 0, \lambda = k; R_2)| \geq \epsilon$, where ϵ is a constant which characterizes the efficiency of detectors. In principle, it even may be that, for some value $\lambda = k$, the set $\{\pi \in \mathcal{E}_1 : \lambda(\pi) = k\} = \emptyset$ and $\{\pi \in \mathcal{E}_2 : \lambda(\pi) = k\} \approx \mathcal{E}_2$. This is the particular case of nonexistence of collectives (fluctuating distributions). Here we also have modified Bell's and CHSH inequalities (8), (9).

(b) Collectives (if they are) $x_{a,\lambda}$ and $x_{b,\lambda}$ may be uncombinable (probability distribution $\mathbf{p}(a = \alpha, \lambda = k, b = \beta)$ may be not exists). Even if $n_N(a \neq 0, \lambda = k)/N$ stabilize, $n_N(a \neq 0, \lambda = k, b \neq 0)/N$ need not stabilize, because a and b may record ensembles with essentially different distributions of λ.

One of rather surprising consequences of our frequency approach is that the

[12] Of course, (1) implies also the existence of FR-numbers $\mathbf{p}_{\alpha\beta\gamma}$.

consideration of variables $a, b, ...$ which can take the value 0 does not solve the problem of the efficiency of detectors (compare with J. Bell [1]).

Conclusion

In the frequency approach (if we follow to R. von Mises and define probabilities as limits of relative frequencies and not as abstract Kolmogorov measures) it seems to be that not locality or determinism play the most important role in Bell's framework. First of all those formal probabilities $p(a = \pm 1, b = \pm 1/\lambda)$ *(which are commonly used) need not exist at all (for frequency approach it is rather normal situation). Secondly BCHS locality condition has not a meaning of locality condition. It was rightly called "outcome independence condition" [3]. However, everybody who works in Kolmogorov's axiomatic approach (conventional probability theory) consider dependence or independence as dependence or independence of EVENTS. Of course, such a viewpoint implies nonlocality: one event depends on another. In von Mises approach dependence or independence has the meaning of dependence or independence of collectives (random sequences). Such a dependence is a consequence of the simultaneous preparation procedure for two collectives. Of course, this does not exclude the possibility that some nonlocal effects also play some role in the creation of such a dependence.*

Appendix 1.

Let us consider the motion of a particle on the line. A preparation procedure Π produces particles with velocities $v = +1$ and $v = -1$. Suppose that Π cannot control (even statistically) proportion of particles moving in positive and negative directions. This proportion fluctuates from run to run. Mathematically we can describe this situation as the absence of the statistical stabilization in the sequence: $x_v = (v_1, v_2, ..., v_N, ...)$, $v_j = \pm 1$, of velocities of particles. For example, let relative frequencies $\nu_N(v = +1) \approx \sin^2 \phi_N$ and $\nu_N(v = -1) \approx \cos^2 \phi_N$. If 'phases' ϕ_N do not stabilize (mod 2π) when $N \to \infty$, then frequencies $\nu_N(v = +1), \nu_N(v = -1)$ fluctuate when $N \to \infty$. Hence the sequence x_v is not a collective (the principle of the statistical stabilization is violated). Suppose that we have an apparatus to measure the energy of a particle: $E = v^2/2$. We obtain that $E = 1/2$ with the probability one. Suppose that we cannot measure the velocity. Then we would not know that the measured value $E = 1/2$ is produced by chaotic fluctuations of the (objective) velocity.

A slight modification can give an example in that 'fluctuating microreality' produces states which are not eigenstates of the E. Let $v = \pm 1, \pm 1/2$ and let $\nu_N(v = +1) = \nu_N(v = -1/2) \approx \frac{1}{2} \sin^2 \phi_N$ and $\nu_N(v = -1) = \nu_N(v = +1/2) \approx$

$\frac{1}{2}\cos^2\phi_N$. Suppose that again 'phases' ϕ_N do not stabilize. Thus probabilities $\mathbf{p}(v = +1), \mathbf{p}(v = -1), \mathbf{p}(v = 1/2), \mathbf{p}(v = -1/2)$ do not exist. However, the frequency probabilities $\mathbf{p}(E = 1/2), \mathbf{p}(E = 1/8)$ are well defined and equal to 1/2. Suppose that we can measure only the energy (and cannot observe this oscillation of probabilities for the velocity). Then we can, in principle, suppose that there exists the probability distribution of the velocity in this experiment and use such a distribution in some considerations. It may be that we do such an illegal trick in Bell's framework.

Appendix 2: Frequency analysis for 'time average model' for the EPR experiment.

The process of a measurement is not a δ-function process. The values of physical observables are time averages of hidden variables λ and $\omega_a, \omega_b, ...$ which evolve with time. In fact, $a = a(\xi_a, \eta_a)$ is a functional of trajectories of the microstates of the apparatus $\xi_a = \omega_a(\cdot)$ and a quantum particle $\eta_a = \lambda_a(\cdot)$. There are the initial conditions $\omega_a(0) = \omega_a^0$ and $\lambda_a(0) = \lambda^0$ (here ω_a^0 is the microstate of a and λ^0 is the value of hidden variable for a quantum particle before the beginning of the interaction). In general we cannot assume that trajectories ξ_a and η_a evolve independently.[13] The interaction between a particle and an apparatus induces the simultaneous evolution of ξ_a and η_a.

Let us consider a series of experiments with correlated particles. For the apparatus a, we have a series of two dimensional trajectories:

$$x_{u_a} = (u_{a1}, u_{a2}, ..., u_{aN}, ...), \; u_j = (\xi_{aj}, \eta_{aj}), \tag{17}$$

where $u_{aj}(t) = (\omega_{aj}(t), \lambda_{aj}(t))$ is the solution of equation:

$$\frac{du_{aj}}{dt} = A_j(u_{aj}(t)), \;\; u(0) = (\omega_a^0, \lambda^0) \; .$$

In general the operator of evolution A depends on the trial j (uncontrolled fluctuations of fields), $A = A_j$. The corresponding series of two dimensional trajectories for the apparatus b is denoted by the symbol x_{u_b}.

We again can ask the question $\mathbf{Q}_1$. Here we have to be more careful with the choice of a label set. Suppose that all trajectories are continuous. Denote by the symbol C the space of continuous trajectories endowed with the uniform norm. Denote by symbol $\mathcal{B}(C)$ the σ-field of Borel subsets of the

[13]Therefore we use index a for the trajectory η_a of the hidden variable λ of a quantum particle.

metric space C. In principle, we are interested in the statistical stabilization of frequencies $\nu_N(u \in A \times B; x_{u_a}) = n_N(u \in A \times B; x_{u_a})/N$, where $A, B \in \mathcal{B}(C)$. However, there is no such a stabilization for Borel sets even in finite dimensional case, see R. von Mises [11]. So there are no reasons to hope that (17) can be a collective with respect to the set of labels

$$L = \{A \times B : A, B \in \mathcal{B}(C)\}.$$

Thus the existence of a Kolmogorov probability distribution $\mathbf{p}(\xi_a \in A, \eta_a \in B)$ on the set of hidden parameters (ξ_a, η_a) is extremely questionable.

There is nothing special in the set of hidden variables which is considered in this section. Of course, this hidden variables (trajectories) belong to an infinite dimensional space. However, as it was continuously underlined, Bell's proof does not depend on the structure of the space of hidden variables. Here the proof is blocked not as a consequence of the special structure of the space of hidden variables, but as a consequence of the absence of a Kolmogorov probability distribution.

Nevertheless, suppose that x_{u_a} is a collective with respect to some subfield $\mathcal{B}_0(C)$ of $\mathcal{B}(C)$. Thus

$$\mathbf{p}(\xi_a \in A, \eta_a \in B) = \lim_{N \to \infty} \nu_N(u \in A \times B; x_{u_a}), \quad A, B \in \mathcal{B}_0(C) \, .$$

In such a case we obtain a collective with respect to the label set $L_0 = \{A \times B : A, B \in \mathcal{B}_0(C)\}$. Here $\mathbf{p}$ is not a Kolmogorov σ-additive measure, but only a finite additive measure. In principle, we cannot proceed Bell's proof.

This proof is also blocked, because collectives x_{ξ_a} and x_{η_a} consisting of trajectories $\omega_a(t)$ and $\lambda_a(t)$, respectively, are not independent. Dependence is generated in the process of evolution via the mixing by the evolution operator A. We do not have the factorization condition: $\mathbf{p}(\xi_a \in A, \eta_a \in B; x_{u_a}) = \mathbf{p}(\xi_a \in A; x_{\xi_a}) \, \mathbf{p}(\eta_a \in B; x_{\eta_a})$

Despite all of these troubles we continue our analysis.

The next question is $\mathbf{Q}_2$. It is clear that we cannot hope to have combining of x_{u_a} and x_{u_b} for the label set $L \times L$. Thus there is no Kolmogorov measure on the σ-field of Borel subsets of $C \times C \times C \times C$. Well, we can in principle assume that they are combinable with respect to the label set $L_0 \times L_0$. Hence there may exist a finite additive measure $\mathbf{p}(\xi_a \in A_1, \eta_a \in B_1, \xi_b \in A_2, \eta_b \in B_2)$. Of course, the absence of σ-additivity is a mathematical trouble. However, the main problem is that collectives x_{u_a} and x_{u_b} are not independent, because trajectories u_a and u_b are connected at the initial instant of time by the constraint: $\lambda_a(0) = \lambda_b(0) = \lambda^0$.

Therefore in the present model there are no doubts in the dependence of collectives corresponding to different measurement apparatuses. There is no factorization of 'probability':

$$\mathbf{p}(\xi_a \in A_1, \eta_a \in B_1, \xi_b \in A_2, \eta_b \in B_2) = \mathbf{p}(\xi_a \in A_1, \eta_a \in B_1)\ \mathbf{p}(\xi_b \in A_2, \eta_b \in B_2)\ .$$

In general there is no Bell's (or CHSH) inequality. Here the dependence of collectives is a trivial consequence of the dependence of initial conditions. [14]

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